

ON FLOW-EQUIVALENCE OF \mathcal{R} -GRAPH SHIFTS

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ABSTRACT. We show that Property (A) of subshifts and the semigroup, that is associated to subshifts with Property (A), are invariants of flow equivalence. We show for certain \mathcal{R} -graphs that their isomorphism is implied by the flow equivalence of their \mathcal{R} -graph shifts.

1. INTRODUCTION

Let Σ be a finite alphabet, and let S_Σ be the shift on the shift space $\Sigma^\mathbb{Z}$,

$$S_\Sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.$$

S_Σ -invariant closed subsets X of $\Sigma^\mathbb{Z}$ (more precisely, with S_X denoting the restriction of S_Σ to X , the dynamical systems (X, S_X) are called subshifts. These are the subject of symbolic dynamics. For an introduction to symbolic dynamics see [Ki] or [LM].

A word is called admissible for a subshift $X \subset \Sigma^\mathbb{Z}$ if it appears in a point of X . We denote the set of admissible words of a subshift $X \subset \Sigma^\mathbb{Z}$ by $\mathcal{L}(X)$. The language $\mathcal{L}(X)$ is factorial and bi-extensible, and every factorial and bi-extensible language is the set of admissible words of a unique subshift.

Let \bullet be a symbol that is not in Σ , and consider a subshift $X \subset \Sigma^\mathbb{Z}$. Denote by $\varphi^{(\sigma)}$ the mapping that assigns to a word $a \in \mathcal{L}(X)$ the word that is obtained from a by carrying out the substitution that replaces the symbol σ by the word $\sigma\bullet$. The set of subwords of the words in $\varphi^{(\sigma)}(\mathcal{L}(X))$ is a factorial and bi-extensible language, and we denote the subshift that it determines by $X^{(\sigma)}$. One says that the subshift $X^{(\sigma)}$ arises from the subshift X by symbol expansion. In Section 2 we describe some effects of symbol expansion.

Subshifts $X \subset \Sigma^\mathbb{Z}$ and $\tilde{X} \subset \tilde{\Sigma}^\mathbb{Z}$ are called flow equivalent if there exists a sequence $Z_k, 1 \leq k \leq K, K \in \mathbb{N}$, of subshifts, such that $X = Z_1$ and $\tilde{X} = Z_K$, and such that Z_k is topologically conjugate to Z_{k-1} , or Z_k is obtained from Z_{k-1} by symbol expansion, or Z_{k-1} is obtained from Z_k by symbol expansion, $1 < k \leq K$. Flow equivalence was introduced by Parry and Sullivan in 1975 [PS]. Next to topological conjugacy it is one of the fundamental equivalence relations for subshifts.

The notions of \mathcal{R} -graph, \mathcal{R} -graph semigroup, and \mathcal{R} -graph shift were introduced in [Kr2]. The class of \mathcal{R} -graph shifts contains the class of Markov-Dyck shifts [M3]. In Section 5 we show for certain \mathcal{R} -graphs, that the flow equivalence of their \mathcal{R} -graph shifts implies their isomorphism. This extends a result of Costa and Steinberg [CS] for Markov-Dyck shifts. The proof uses Property (A) and the semigroup that is associated to subshifts with Property (A) [Kr1]. In Section 3 we prove invariance under flow equivalence of Property (A) and in Section 4 we prove invariance under flow equivalence of the associated semigroup. For an extension of the theory beyond subshifts with Property (A) see Costa and Steinberg [CS].

In Section 5 we consider \mathcal{R} -graph shifts. In [HK] there was given a necessary and sufficient condition for an \mathcal{R} -graph to have an \mathcal{R} -graph shift with Property (A), whose associated semigroup is the \mathcal{R} -graph semigroup of the \mathcal{R} -graph. Under

this condition we prove in Section 5, that the flow equivalence of \mathcal{R} -graph shifts implies the isomorphism of the underlying \mathcal{R} -graphs.

2. SYMBOL EXPANSION

We introduce notation for subshifts $X \subset \Sigma^{\mathbb{Z}}$. The S_X -orbit of a point $x \in X$ we denote by $O_X(x)$, and for an S_X -invariant set $A \subset X$ we denote the set of S_X -orbits in A by $\Omega(A)$. The period of a periodic point $p \in X$ we denote by $\pi(p)$. We set

$$x_{[i,j]} = (x_k)_{i \leq k \leq j},$$

and

$$X_{[i,j]} = \{x_{[i,j]} : x \in X\}, \quad i, j \in \mathbb{Z}, i \leq j, \quad x \in X,$$

and we use similar notation in the case that indices range in semi-infinite intervals. (The elements in $X_{[i,j]}$, $X_{[i,\infty)}$, $X_{(\infty,i]}$ can be identified with the words they carry. From the context it becomes clear, if such an identification is made.) We set

$$\Gamma_X^+(a) = \{x^+ \in X_{(j,\infty)} : ax^+ \in X_{[i,\infty)}\}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$

The notation Γ^- has the symmetric meaning. We also set

$$\omega_X^+(a) = \bigcap_{x^- \in \Gamma^-(a)} \{x^+ \in \Gamma^+(a) : x^- ax^+ \in X\}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$

The notation ω^- has the symmetric meaning. And we set

$$\Gamma_X(a) = \{(x^-, x^+) \in \Gamma^-(a) \times \Gamma^+(a) : x^- ax^+ \in X\}, \quad a \in x_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$

Let $\sigma \in \Sigma$, let \bullet be a symbol that is not in Σ , and consider for a subshifts $X \subset \Sigma^{\mathbb{Z}}$ the subshift $X^{(\sigma)} \subset (\Sigma \cup \{\bullet\})^{\mathbb{Z}}$. We denote by $\varphi_-^{(\sigma)}(\varphi_+^{(\sigma)})$ the mapping that assigns to $x^- \in X_{(-\infty,0)}(x^+ \in X_{[0,\infty)})$ the point $n \in X_{(-\infty,0)}^{(\sigma)}(X_{[0,\infty)}^{(\sigma)})$ that is obtained from $x^-(x^+)$ by carrying out the substitution that replaces the symbol σ by the word $\sigma\bullet$. Also we denote by $\varphi^{(\sigma)}$ the mapping that assigns to a point $x \in X$ the point in $X^{(\sigma)}$, that is given by

$$\varphi^{(\sigma)}(x)_{(-\infty,0)} = \varphi_-^{(\sigma)}(x_{(-\infty,0)}), \quad \varphi^{(\sigma)}(x)_{[0,\infty)} = \varphi_+^{(\sigma)}(x_{[0,\infty)}).$$

One observes that

$$\varphi^{(\sigma)}(O_X(x)) \subset O_{X^{(\sigma)}}(\varphi^{(\sigma)}(x)), \quad x \in X.$$

For precision we note, that one has, with $\ell^-(x, n)(\ell^+(x, n))$ denoting the length of $\varphi^{(\sigma)}(x)_{[-n,0)}(\varphi^{(\sigma)}(x)_{[0,n)})$, that

$$\begin{aligned} \varphi^{(\sigma)}(S_X^{-n}(x)) &= S_{X^{(\sigma)}}^{-\ell^-(x,n)}(\varphi^{(\sigma)}(x)), \\ \varphi^{(\sigma)}(S_X^{-n}(x)) &= S_{X^{(\sigma)}}^{-\ell^-(x,n)}(\varphi^{(\sigma)}(x)), \quad n \in \mathbb{N}. \end{aligned}$$

Also,

$$\varphi^{(\sigma)}(X) \cup S_{X^{(\sigma)}}(\varphi^{(\sigma)}(X)).$$

We denote the bijection of $\Omega(X)$ onto $\Omega(X^{(\sigma)})$ that assigns to the S_X -orbit of $x \in X$ the $S_{X^{(\sigma)}}$ -orbit of $\varphi^{(\sigma)}(x)$ by ξ_σ .

Lemma 2.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma, a \in \mathcal{L}(X)$, one has*

$$\varphi_+^{(\sigma)}(\omega_X^+(a)) = \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a)).$$

Proof. We prove that $\varphi^{(\sigma)}(\omega_X^+(a)) \subset \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a))$. Let $x^+ \in \omega_X^+(a)$, and let

$$y^- \in \Gamma_{X^{(\sigma)}}^-(\varphi^{(\sigma)}(a)).$$

It follows from $\varphi^{(\sigma)}(a)_0 \neq \bullet$, that $y_{-1}^- \neq \sigma$, and one sees that y^- is in the image of $\varphi_-^{(\sigma)}$. Its inverse image x^- under $\varphi_-^{(\sigma)}$ is in $\Gamma_X^-(a)$. It follows that $x^-ax^+ \in X$, and therefore

$$\varphi^{(\sigma)}(x^-ax^+) = y^- \varphi^{(\sigma)}(a) \varphi_+^{(\sigma)}(x^+) \in X^{(\sigma)},$$

and this means that $\varphi_+^{(\sigma)}(x^+) \in \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a))$.

For the converse one has a similar argument. \square

Lemma 2.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma, b, b' \in \mathcal{L}(X)$, one has*

$$\Gamma_X(b) = \Gamma_X(b'),$$

if and only if

$$\Gamma_{X^{(\sigma)}}(\varphi^{(\sigma)}(b)) = \Gamma_{X^{(\sigma)}}(\varphi^{(\sigma)}(b')).$$

Proof. The lemma follows from

$$\Gamma_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a)) \subset \varphi_+^{(\sigma)}(\Gamma_X^+(a)), \quad \Gamma_{X^{(\sigma)}}^-(\varphi^{(\sigma)}(a)) \subset \varphi_-^{(\sigma)}(\Gamma_X^-(a)), \quad a \in \mathcal{L}(X). \quad \square$$

3. PROPERTY (A)

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we define a subshift of finite type $A_n(X)$ by

$$A_n(X) =$$

$$\bigcap_{i \in \mathbb{Z}} (\{x \in X : x_{[i, \infty)} \in \omega_X^+(x_{[-n, i]})\} \cap \{x \in X : x_{(-\infty, i]} \in \omega_X^-(x_{[i, i+n]})\}), \quad n \in \mathbb{N},$$

and we set

$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

Lemma 3.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$, and for $\sigma \in \Sigma$, one has*

$$(1) \quad \xi_\sigma(\Omega(A_n(X))) \subset \Omega(A_{2n}(X^{(\sigma)})), \quad n \in \mathbb{N},$$

and

$$(2) \quad \xi_\sigma^{-1}(\Omega(A_n(X^{(\sigma)}))) \subset \Omega(A_n(X)), \quad n \in \mathbb{N}.$$

Proof. We show (1). Let $n \in \mathbb{N}$, let $x \in A_n(X)$, and let $i \in \mathbb{Z}$. Let μ be the number of times that the symbol \bullet appears in $\varphi^{(\sigma)}(x)_{[i, i+2n]}$. Assume that neither $x_i^{(\sigma)} = \bullet$, nor $x_{i+2n-1}^{(\sigma)} = \sigma$. Then

$$\varphi^{(\sigma)}(x_{[i, i+2n-\mu]}) = \varphi^{(\sigma)}(x)_{[i, i+2n]}.$$

From

$$x_{[i+2n-\mu, \infty)} \in \omega_X^+(x_{[i, i+2n-\mu]}),$$

it follows then by Lemma 2.1, that

$$(3) \quad \varphi^{(\sigma)}(x)_{[i+2n, \infty)} \in \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(x)_{[i, i+2n]}).$$

In the case that $x_i^{(\sigma)} = \bullet$, necessarily $x_{i-1}^{(\sigma)} = \sigma$, and in the case that $x_{i+2n-1}^{(\sigma)} = \sigma$, necessarily $x_{i+2n}^{(\sigma)} = \bullet$, and in both cases it is seen that (3) also holds.

For (2) one has a similar argument. \square

We recall from [Kr1] the definition of Property (A). For $n \in \mathbb{N}$ a subshift $X \subset \Sigma^{\mathbb{Z}}$, has property (a, n, H) , $H \in \mathbb{N}$, if for $h, \tilde{h} \geq 3H$ and for $I_-, I_+, \tilde{I}_-, \tilde{I}_+ \in \mathbb{Z}$, such that

$$I_+ - I_-, \tilde{I}_+ - \tilde{I}_- \geq 3H,$$

and for

$$a \in A_n(X)_{(I_-, I_+]}, \quad \tilde{a} \in A_n(X)_{(\tilde{I}_-, \tilde{I}_+]},$$

such that

$$a_{(I_-, I_- + H]} = \tilde{a}_{(\tilde{I}_-, \tilde{I}_- + H]}, \quad a_{(I_+ - H, I_+]} = \tilde{a}_{(\tilde{I}_+ - H, \tilde{I}_+]},$$

one has that

$$\Gamma_X(a) = \Gamma_X(\tilde{a}).$$

It is assumed, that $A(X) \neq \emptyset$. The subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (A) if there are $H_n, n \in \mathbb{N}$, such that X has the properties (a, n, H_n) , $n \in \mathbb{N}$.

Theorem 3.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma$, one has that X has Property (A) if and only if $X^{(\sigma)}$ has Property (A).*

Proof. The theorem follows from Lemma 2.2 and Lemma 3.1. \square

4. THE ASSOCIATED SEMIGROUP

Consider a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property (A). We denote the set of periodic points in $A(X)$ by $P(A(X))$. We introduce a preorder relation $\succsim(X)$ into the set $P(A(X))$ where for $q, r \in P(A(X))$, $q \succsim(X) r$, means that there exists a point in $A(X)$ that is left asymptotic to the orbit of q and right asymptotic to the orbit of r . The equivalence relation on $P(A(X))$ that results from the preorder relation $\succsim(X)$ we denote by $\approx(X)$. We denote the set of $\approx(X)$ -equivalence classes by $\mathfrak{P}(X)$.

Lemma 4.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$, for $\sigma \in \Sigma$. $q, r \in P(A(X))$, and for $\sigma \in \Sigma$, one has*

$$q \succsim(X) r,$$

if and only if

$$\varphi^{(\sigma)}(q) \succsim(X^{(\sigma)}) \varphi^{(\sigma)}(r).$$

Proof. This follows from Lemma 3.1. \square

We recall the construction of the associated semigroup. For a property (A) subshift $X \subset \Sigma^{\mathbb{Z}}$ we denote by $Y(X)$ the set of points in X that are left asymptotic to a point in $P(A(X))$ and also right-asymptotic to a point in $P(A(X))$. Let $y, \tilde{y} \in Y(X)$, let y be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let \tilde{y} be left asymptotic to $\tilde{q} \in P(A(X))$ and right asymptotic to $\tilde{r} \in P(A(X))$. Given that X has the properties (a, n, H_n) , $n \in \mathbb{N}$, we say that y and \tilde{y} are equivalent, $y \approx(X) \tilde{y}$, if $q \approx(X) \tilde{q}$ and $r \approx(X) \tilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \tilde{q}, \tilde{r} \in P(A_n(X))$ and for $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}$, $I < J, \tilde{I} < \tilde{J}$, such that

$$y_{(-\infty, I]} = q_{(-\infty, 0]}, \quad y_{(J, \infty)} = r_{(0, \infty)},$$

$$\tilde{y}_{(-\infty, \tilde{I}]} = \tilde{q}_{(-\infty, 0]}, \quad \tilde{y}_{(\tilde{J}, \infty)} = \tilde{r}_{(0, \infty)},$$

one has for $h \geq 3H_n$ and for

$$a \in X_{(I-h, J+h]}, \quad \tilde{a} \in X_{(\tilde{I}-h, \tilde{J}+h]},$$

such that

$$a_{(I-H_n, J+H_n]} = y_{(I-H_n, J+H_n]}, \quad \tilde{a}_{(\tilde{I}-H_n, \tilde{J}+H_n]} = \tilde{y}_{(\tilde{I}-H_n, \tilde{J}+H_n]},$$

$$a_{(I-h, I-h+H_n)} = \tilde{a}_{(\tilde{I}-h, \tilde{I}-h+H_n)},$$

$$a_{(J+h-H_n, J+h]} = \tilde{a}_{(\tilde{J}+h-H_n, \tilde{J}+h]},$$

and such that

$$a_{(I-h, I]} \in A_n(X)_{(I-h, I]}, \quad \tilde{a}_{(\tilde{J}-h, \tilde{I}]} \in A_n(X)_{(\tilde{J}-h, \tilde{I}]},$$

$$a_{(J, J+h]} \in A_n(X)_{(J, J+h]}, \quad \tilde{a}_{(\tilde{J}, \tilde{J}+h]} \in A_n(X)_{(\tilde{J}, \tilde{J}+h]},$$

that

$$\Gamma_X(a) = \Gamma_X(\tilde{a}).$$

To give $[Y(X)]_{\approx(X)}$ the structure of a semigroup, let $u, v \in Y(X)$, let u be right asymptotic to $q \in P(A(X))$ and let v be left asymptotic to $r \in P(A(X))$. If here $q \gtrsim (X)r$, then $[u]_{\approx(X)} [v]_{\approx(X)}$ is set equal to $[y]_{\approx(X)}$, where y is any point in Y such that there are $n \in \mathbb{N}, I, J, \hat{I}, \hat{J} \in \mathbb{Z}, I < J, \hat{I} < \hat{J}$, such that $q, r \in A_n(X)$, and such that

$$u_{(I, \infty)} = q_{(I, \infty)}, \quad v_{(-\infty, J]} = r_{(-\infty, J]},$$

$$y_{(-\infty, \hat{I}+H_n]} = u_{(-\infty, I+H_n]}, \quad y_{(\hat{J}-H_n, \infty)} = v_{(J-H_n, \infty)},$$

and

$$y_{(\hat{I}, \hat{J}]} \in A_n(X)_{(\hat{I}, \hat{J}]},$$

provided that such a point y exists. If such a point y does not exist, $[u]_{\approx(X)} [v]_{\approx(X)}$ is equal to zero. Also, in the case that one does not have $q \gtrsim (X)r$, $[u]_{\approx(X)} [v]_{\approx(X)}$ is equal to zero.

Consider a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property (A). For $\mathfrak{p} \in \mathfrak{P}(X)$ we choose a $d^{(\mathfrak{p})} \in \mathfrak{p}$, and we set

$$\mathcal{D} = \{d^{(\mathfrak{p})} : \mathfrak{p} \in \mathfrak{P}(X)\}.$$

In order to facilitate the proof of its invariance under flow equivalence we give an alternate description of the semigroup that is associated to X in terms of the system $\mathcal{D} \subset Y_X$ of representatives of the equivalence relation $\approx(X)$. For $y \in O_X(d^{(\mathfrak{p})})$, $\mathfrak{p} \in \mathfrak{P}(X)$, we define a $J(y, d^{(\mathfrak{q})}) \in \mathbb{Z}$ by

$$S_X^{-J(y, d^{(\mathfrak{p})})}(y) = d^{(\mathfrak{p})}, \quad 0 \leq \pi(d^{(\mathfrak{p})}) < \pi(d^{(\mathfrak{p})}).$$

For $\mathfrak{p} \in \mathfrak{P}(X)$ we set

$$H(d^{(\mathfrak{p})}) = \min \{H \in \mathbb{N} : \Gamma_X(\mathfrak{p}_{[0, H\pi(\mathfrak{p})]}) = \Gamma_X(\mathfrak{p}_{[0, (H+1)\pi(d^{(\mathfrak{p})})]})\}.$$

We denote by $Y_X^-(\mathcal{D})$, the set of points in Y_X , that are left asymptotic to the orbit of a point in \mathcal{D} , and also right asymptotic to the orbit of a point in \mathcal{D} . More precisely, we denote by $Y_X^-(d^{(\mathfrak{p})})(Y_X^+(d^{(\mathfrak{p})}))$, the set of points in Y_X , that are left (right) asymptotic to the orbit of $d^{(\mathfrak{p})}$, $\mathfrak{p} \in \mathfrak{P}(X)$. For

$$y \in Y_X^-(d^{(\mathfrak{q})}) \cap Y_X^+(d^{(\mathfrak{r})}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}(X),$$

we set

$$I^-(y) = \begin{cases} J(y, d^{(\mathfrak{q})}), & \text{if } y \in O_X(d^{(\mathfrak{q})}), \\ \max\{I \in \mathbb{Z} : y_{(-\infty, I]} = d_{(-\infty, 0]}^{(\mathfrak{q})}\}, & \text{if } y \notin O_X(d^{(\mathfrak{q})}), \end{cases}$$

$$I^+(y) = \begin{cases} J(y, d^{(\mathfrak{r})}), & \text{if } y \in O_X(d^{(\mathfrak{r})}), \\ \min\{I \in \mathbb{Z} : y_{[I, \infty)} = d_{[0, \infty)}^{(\mathfrak{r})}\}, & \text{if } y \notin O_X(d^{(\mathfrak{r})}). \end{cases}$$

We say that $O, O' \in \Omega(Y_X^{(D)})$ are $\approx(D)$ -equivalent, if O and O' are left asymptotic to the same periodic orbit, and also right asymptotic to the same periodic orbit, and if, with $\mathfrak{q} \in \mathfrak{P}$ such that y and y' are right asymptotic to the orbit of $d^{(\mathfrak{q})}$ and with

$\mathfrak{r} \in \mathfrak{P}$ such that y and y' are left asymptotic to the orbit of $d^{(\mathfrak{r})}$, there exist $y \in O$ and $y' \in O'$ such that

$$\Gamma_X(d_{[0, H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})})}^{(\mathfrak{q})})y_{[I^-(y), I^+(y)]}d_{[0, H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})})}^{(\mathfrak{r})}) = \Gamma_X(d_{[0, H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})})}^{(\mathfrak{q})})y'_{[I^-(y'), I^+(y')] }d_{[0, H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})})}^{(\mathfrak{r})}).$$

To give $\Omega(Y_X^{(D)})$ the structure of a semigroup, let $\mathfrak{q}, \mathfrak{p}, \mathfrak{r} \in \mathfrak{P}$, and let for points

$$u \in Y_X^-(\mathfrak{q}) \cap Y_X^+(\mathfrak{p}), \quad v \in Y_X^-(\mathfrak{p}) \cap Y_X^+(\mathfrak{r}),$$

in case, that the word

$$(3) \quad d_{[0, H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})})}^{(\mathfrak{q})})y_{[I^-(u), I^+(u)]}d_{[0, H(d^{(\mathfrak{p})})\pi(d^{(\mathfrak{p})})}^{(\mathfrak{p})})y_{[I^-(v), I^+(v)]}d_{[0, H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})})}^{(\mathfrak{r})},$$

is admissible for X , let a point $y[u, v] \in Y_X^-(\mathfrak{q}) \cap Y_X^+(\mathfrak{r})$ be given by

$$y[u, v]_{(-\infty, 0)} = d_{(-\infty, 0)}^{(\mathfrak{q})},$$

and

$$y[u, v]_{[0, \infty)} = d_{[0, H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})})}^{(\mathfrak{q})})y_{[I^-(u), I^+(u)]}d_{[0, H(d^{(\mathfrak{p})})\pi(d^{(\mathfrak{p})})}^{(\mathfrak{p})})y_{[I^-(v), I^+(v)]}d_{[0, \infty)}^{(d^{(\mathfrak{r})})}.$$

and set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = [O(y[u, v])]_{\approx(D)}.$$

In case, that the word (3) is not admissible for X , set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = 0.$$

Also, for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, if

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}) \neq \emptyset,$$

define a $\approx(D)$ -equivalence class $\gamma(\mathfrak{q}, \mathfrak{r})$ by

$$\gamma(\mathfrak{q}, \mathfrak{r}) = [O(y)]_{\approx(D)}, \quad y \in Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}).$$

As a consequence of Property (A) of X the $\approx(D)$ -equivalence class $\gamma(\mathfrak{q}, \mathfrak{r})$ is well defined. If

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}) = \emptyset,$$

set

$$\gamma(\mathfrak{q}, \mathfrak{r}) = 0.$$

Identify $\mathfrak{p} \in \mathfrak{P}$ with $\gamma(\mathfrak{p}, \mathfrak{p})$. Finally for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, and for $u \in Y_X^+(\mathfrak{q}), v \in Y_X^-(\mathfrak{r})$, set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = [u]_{\approx(D)}\gamma(\mathfrak{q}, \mathfrak{r})[v]_{\approx(D)}.$$

An isomorphism $\eta_{\sigma, D}$ of $[Y_X]_{\approx(X)}$ onto $[\Omega(Y_X^{(D)})]_{\approx(D)}$ is obtained by choosing out of every $\approx(X)$ -equivalence class α a point $\eta(\alpha) \in Y_X^{(D)}$, and by setting

$$\eta_X^{(D)}(\alpha) = [\eta(\alpha)]_{\approx(D)}.$$

Theorem 4.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property (A) and for $\sigma \in \Sigma$, the semigroups, that are associated to X and $X^{(\sigma)}$, are isomorphic.*

Proof. Set

$$d^{(\mathfrak{p}^{(\sigma)})} = \varphi^{(\sigma)}(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P}.$$

One has

$$\pi(d^{(\mathfrak{p}^{(\sigma)})}) = \pi(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P},$$

and one has by Lemma 2.2, that

$$H(d^{(\mathfrak{p}^{(\sigma)})}) = H(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P}.$$

Setting

$$D^{(\sigma)} = \{d^{(\mathfrak{p}^{(\sigma)})} : \mathfrak{p} \in \mathfrak{P}\}.$$

yields a system of representatives of the $\approx (X^{(\sigma)})$ -equivalence classes in $\mathfrak{P}(X^{(\sigma)})$. By construction

$$\varphi^{(\sigma)}(y[u, v]) = y[\varphi^{(\sigma)}(u), \varphi^{(\sigma)}(v)], \quad u, v \in Y_X.$$

Also, by Lemma 3.1, for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$,

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^-(\mathfrak{r}) \neq \emptyset,$$

if and only if

$$Y_X^-(\mathfrak{q}^{(\sigma)}) \cap A(X^{(\sigma)}) \cap Y_X^-(\mathfrak{r}^{(\sigma)}) \neq \emptyset.$$

It follows that an isomorphism $\psi_{\sigma, D}$ of $[Y_X^{(D)}]_{\approx(D)}$ onto $[Y_{X^{(\sigma)}}^{(D^{(\sigma)})}]_{\approx(D^{(\sigma)})}$ is given by setting

$$\psi_{\sigma, D}([y]) = [\varphi^{(\sigma)}(y)], \quad y \in Y_X^{(D)},$$

and one obtains an isomorphism $\Xi^{(\sigma)}$ of $[Y_X]_{\approx(X)}$ onto $[Y_{X^{(\sigma)}}]_{\approx(X^{(\sigma)})}$ by setting

$$\Xi^{(\sigma)} = \eta_{\sigma, D}^{-1} \psi_{\sigma, D} \eta_{\sigma, D}. \quad \square$$

For the invariance of the associated semigroup under flow equivalence, under the assumption that $A(X)$ is dense in X , or in the sofic case, see also [CS, Theorem 9.20]).

The semigroup $[Y_X^{(D)}]_{\approx(D)}$ is a set of equivalence classes of orbits. As originally done in [Kr1], we have introduced the associated semigroup of a subshift with Property (A) in terms of equivalence classes of points, rather than equivalence classes of orbits. However, since points in Y_X , that are in the same orbit, are $\approx (X)$ -equivalent, one can define the associated semigroup in the first place as a set of equivalence classes of orbits. The same remark applies to the set of idempotents \mathfrak{P} . When the associated semigroup is introduced as a set of equivalence classes of orbits, then the mapping ξ_σ is seen to induce the isomorphism of the associated semigroup of X onto the associated semigroup of $X^{(\sigma)}$.

5. \mathcal{R} -GRAPH SHIFTS

Given finite sets \mathcal{E}^- and \mathcal{E}^+ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we set

$$\mathcal{E}^-(\mathcal{R}) = \{e^- \in \mathcal{E}^- : \{e^-\} \times \mathcal{E}^+ \subset \mathcal{R}\}, \quad \mathcal{E}^+(\mathcal{R}) = \{e^+ \in \mathcal{E}^+ : \mathcal{E}^- \times \{e^+\} \subset \mathcal{R}\}.$$

and

$$\Omega_{\mathcal{R}}^+(e^-) = \{e^+ \in \mathcal{E}^+ : (e^-, e^+) \in \mathcal{R}\}, \quad e^- \in \mathcal{E}^-,$$

$$\Omega_{\mathcal{R}}^-(e^+) = \{e^- \in \mathcal{E}^- : (e^-, e^+) \in \mathcal{R}\}, \quad e^+ \in \mathcal{E}^+.$$

We recall from [Kr2] the notion of an \mathcal{R} -graph. Let there be given a finite directed graph with vertex set \mathfrak{P} and edge set \mathcal{E} . Assume also given a partition

$$\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+.$$

With s and t denoting the source and the target vertex of a directed edge we set

$$\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) = \{e^- \in \mathcal{E}^- : s(e^-) = \mathfrak{q}, t(e^-) = \mathfrak{r}\},$$

$$\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) = \{e^+ \in \mathcal{E}^+ : s(e^+) = \mathfrak{r}, t(e^+) = \mathfrak{q}\}, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

We assume that $\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$ if and only if $\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, and we assume that the directed graph $(\mathfrak{P}, \mathcal{E}^-)$ is strongly connected, or, equivalently, that the directed graph $(\mathfrak{P}, \mathcal{E}^+)$ is strongly connected. Let there further be given relations

$$\mathcal{R}(\mathfrak{q}, \mathfrak{r}) \subset \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \times \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

and set

$$\mathcal{R} = \bigcup_{q, r \in \mathfrak{P}} \mathcal{R}(q, r).$$

The resulting structure, for which we use the notation $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, is called an \mathcal{R} -graph.

We also recall the construction of a semigroup (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ from an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as described in [Kr2]. The semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ contains idempotents $\mathbf{1}_p, p \in \mathfrak{P}$, and has \mathcal{E} as a generating set. Besides $\mathbf{1}_p^2 = \mathbf{1}_p, p \in \mathfrak{P}$, the defining relations are:

$$f^- g^+ = \mathbf{1}_q, \quad f^- \in \mathcal{E}^-(q, r), g^+ \in \mathcal{E}^+(q, r), (f^-, g^+) \in \mathcal{R}(q, r), \quad q, r \in \mathfrak{P},$$

and

$$\begin{aligned} \mathbf{1}_q e^- &= e^- \mathbf{1}_r = e^-, & e^- &\in \mathcal{E}^-(q, r), \\ \mathbf{1}_r e^+ &= e^+ \mathbf{1}_q = e^+, & e^+ &\in \mathcal{E}^+(q, r), \quad q, r \in \mathfrak{P}, \end{aligned}$$

$$f^- g^+ = \begin{cases} \mathbf{1}_q, & \text{if } (f^-, g^+) \in \mathcal{R}(q, r), \\ 0, & \text{if } (f^-, g^+) \notin \mathcal{R}(q, r), \quad f^- \in \mathcal{E}^-(q, r), g^+ \in \mathcal{E}^+(q, r), \quad q, r \in \mathfrak{P}, \end{cases}$$

and

$$\mathbf{1}_q \mathbf{1}_r = 0, \quad q, r \in \mathfrak{P}, q \neq r.$$

The semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is called an \mathcal{R} -graph semigroup.

The \mathcal{R} -graph shift $MD_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ of the \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is the subshift

$$MD_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+) \subset (\{\mathcal{E}^- \cup \mathcal{E}^+\}^{\mathbb{Z}})$$

with the admissible words $(\sigma_i)_{1 \leq i \leq I}, I \in \mathbb{N}$, of $MD_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ given by the condition

$$\prod_{1 \leq i \leq I} \sigma_i \neq 0.$$

For an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we denote by $\mathfrak{P}^{(1)}$ the set of vertices in \mathfrak{P} that have a single predecessor vertex in \mathcal{E}^- , or, equivalently, that have a single successor vertex in \mathcal{E}^+ . For $p \in \mathfrak{P}^{(1)}$ the predecessor vertex of p in \mathcal{E}^- , which is identical to the successor vertex of p in \mathcal{E}^+ , is denoted by $\kappa(p)$. We set

$$\mathcal{E}_{\mathcal{R}}^- = \bigcup_{p \in \mathfrak{P}^{(1)}} \mathcal{E}^-(\mathcal{R}(\kappa(p), p)), \quad \mathcal{E}_{\mathcal{R}}^+ = \bigcup_{p \in \mathfrak{P}^{(1)}} \mathcal{E}^+(\mathcal{R}(\kappa(p), p)),$$

and

$$\mathfrak{P}_{\mathcal{R}}^{(1)} = \{p \in \mathfrak{P}^{(1)} : \mathcal{R}(\kappa(p), p) = \mathcal{E}^-(\kappa(p), p) \times \mathcal{E}^+(\kappa(p), p)\}.$$

We formulate conditions (a), (b), (c) and (d) on an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as follows:

$$(a-) \quad \Omega_{\mathcal{R}(q, r)}^+(e^-) \neq \Omega^+(\tilde{e}^-), \quad e^-, \tilde{e}^- \in \mathcal{E}^-(q, r), e^- \neq \tilde{e}^-, \quad q, r \in \mathfrak{P},$$

$$(a+) \quad \Omega_{\mathcal{R}(q, r)}^-(e^+) \neq \Omega^-(\tilde{e}^+), \quad e^+, \tilde{e}^+ \in \mathcal{E}^+(r, q), e^+ \neq \tilde{e}^+, \quad q, r \in \mathfrak{P}.$$

$$(b-) \quad \text{There is no non-empty cycle in } \mathcal{E}_{\mathcal{R}}^-,$$

$$(b+) \quad \text{There is no non-empty cycle in } \mathcal{E}_{\mathcal{R}}^+,$$

$$(c) \quad \text{For } p \in \mathfrak{P}^{(1)} \text{ such that } \kappa(p) \neq p, \mathcal{E}_{\mathcal{R}}^-(p) = \emptyset, \text{ or } \mathcal{E}_{\mathcal{R}}^-(p) = \emptyset,$$

$$(d) \quad \text{For } q, r \in \mathfrak{P}^{(1)}, q \neq r, \text{ there do not simultaneously exist a path in } \mathcal{E}_{\mathcal{R}}^- \text{ from } q \text{ to } r \text{ and a path in } \mathcal{E}_{\mathcal{R}}^+ \text{ from } q \text{ to } r,$$

Theorem 5.1. *For \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfy the Conditions (a), (b), (c) and (d) the flow equivalence of the \mathcal{R} -graph shifts $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.*

Proof. By Theorem 2.3 of [HK] and Theorem 6.1 of [HK] the conditions imply that the \mathcal{R} -graph shift $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ has property (A), and that the semigroup, that is associated to it, is $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. By Theorem 4.2 the flow equivalence of the shifts $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the \mathcal{R} -graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, which in turn, by Theorem 2.1 of [Kr2], implies the isomorphism of the \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. \square

Theorem 5.1 extends the result of Costa and Steinberg, that the flow equivalence of Markov-Dyck shifts of finite irreducible directed graphs, in which every vertex has at least two incoming edges, implies the isomorphism of the graphs (see [CS, Theorem 8.6]),

Let for $K > 1$, B_K denote the full shift on K symbols, and let D_2 denote the Dyck shift on four symbols. The shifts $D_2 \times B_K, K > 1$, belong to the class of \mathcal{R} -graph shifts. They arise from the one-vertex \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\{\mathfrak{p}\}, \mathcal{E}^-, \mathcal{E}^+)$, where

$$\mathcal{E}^- = \{e^-(m, \beta) : 1 \leq m \leq K, \beta = 0, 1\}, \quad \mathcal{E}^+ = \{e^+(l, \beta) : 1 \leq l \leq K, \beta = 0, 1\},$$

and where

$$(e^-(m, \beta^-), e^-(l, \beta^+)) \in \mathcal{R}(\mathfrak{p}, \mathfrak{p}),$$

if and only if

$$\beta^- = \beta^+, \quad 1 \leq m \leq K, 1 \leq l \leq K.$$

These \mathcal{R} -graphs do not satisfy the conditions of Theorem 5.1, but the \mathcal{R} -graph shifts $D_2 \times B_K, K > 1$, have Property (A), and the flow equivalence of their \mathcal{R} -graph shifts $D_2 \times B_K, K > 1$, still implies the isomorphism of these \mathcal{R} -graphs. This can be seen from the invariance under flow equivalence of the K-groups of subshifts as shown by Matsumoto in [M1], and from

$$K_0(D_2 \times B_K) = \mathbb{Z}[\frac{1}{n}]^\infty, \quad K > 1,$$

as also shown by Matsumoto [M2, Section 8]. Note that the associated semigroup of $D_2 \times B_K, K > 1$, is the Dyck inverse monoid \mathcal{D}_2 .

REFERENCES

- [CS] A. COSTA AND B. STEINBERG, *A categorical invariant of flow equivalence of shifts*, arXiv: 1304.3487 [math.DS]
- [HK] T. HAMACHI AND W. KRIEGER, *A construction of subshifts and a class of semigroups*, arXiv: 1303.4158 [math.DS]
- [Ki] B. P. KITCHENS, *Symbolic dynamics*, Springer, Berlin, Heidelberg, New York (1998)
- [Kr1] W. KRIEGER, *On a syntactically defined invariant of symbolic dynamics*, Ergod. Th. & Dynam. Sys. **20** (2000), 501 – 516
- [Kr2] W. KRIEGER, *On subshift presentations*, arXiv: 1209.2578 [math.DS]
- [LM] D. LIND AND B. MARCUS, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995)
- [M1] K. MATSUMOTO, *Bowen-Franks groups as an invariant for flow equivalence of subshifts*, Ergod. Th. & Dynam. Sys. **21** (2001), 1831 – 1842
- [M2] K. MATSUMOTO, *K-theoretic invariants and conformal measures of the Dyck shifts*, International J. of Mathematics **16** (2005), 213 – 248
- [M3] K. MATSUMOTO, *C*-algebras arising from Dyck systems of topological Markov chains*, Math. Scand. **109** (2011), 31 – 54
- [PS] B. PARRY AND D. SULLIVAN, *A topological invariant for flows on one-dimensional spaces*, Topology **14** (1975), 297 – 299

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